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THE SLIMMEST GEOMETRIC LATTICES

BY

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ABSTRACT. The Whitney numbers of a finite geometric lattice L of rank r are the numbers W_k of elements of rank k and the coefficients w_k of the characteristic polynomial of L, for $0 \le k \le r$. We establish the following lower bounds for the W_k and the absolute values $w_k^+ = (-1)^k w_k$ and describe the lattices for which equality holds (nontrivially) in each case:

$$W_k \ge \binom{r-2}{k-1}(n-r) + \binom{r}{k}, \quad w_k^+ \ge \binom{r-1}{k-1}(n-r) + \binom{r}{k},$$

where $n = W_1$ is the number of points of L.

1. Introduction. Let L be a finite geometric lattice of rank r. The numbers $W_1, W_2, W_3, \cdots, W_{r-1}$ of points, lines, planes, $\cdots$, copoints of L are its *Whitney numbers of the second kind.* Rota has conjectured that the sequence $\{W_k\}$ is *unimodal*: $W_j \ge \min\{W_i, W_k\}$ when $i \le j \le k$, a property known to hold for the partition lattices [8], [9], matroid designs [11], and all geometric lattices with eight or fewer points [3]. A related conjecture is $W_{r-k} \ge W_k$ when $k \le r/2$, which holds with equality for modular geometric lattices. The latter has been proved in the case $k = 1$ [1], [6], and the equality $W_{r-1} = W_1$ shown to characterize modular geometric lattices [6]. Apparently little else is known in general about the sequence $\{W_k\}$, except that $W_k \ge W_1$ for $1 \le k \le r-1$, which follows easily from $W_{r-1} \ge W_1$.(3) We prove here (Theorem 1) a lower bound for W_k in terms of k, r, and the number $n = W_1$ of points of L, which improves the above bound substantially when $2 \le k \le r-2$. Our bound is attained only by the direct product of a modular plane and a free geometry when $2 \le k \le r-2$, and for these lattices equality holds for all k. In view of the representation of a finite lattice by its Hasse diagram, the term "slimmest" is accordingly a fitting description of these extremal lattices.

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(3) In [5] we prove $W_1 + W_2 + \cdots + W_k \le W_{r-k} + \cdots + W_{r-2} + W_{r-1}$, for all k.

The integers $w_k = \Sigma \mu(0, x)$, the sum over all x of rank k in a geometric lattice L with Möbius function μ, are its *Whitney numbers* of the first kind. These are the coefficients in the characteristic (or chromatic) *polynomial* of L, of importance in the *critical problem* [4]. A well-known conjecture in graph theory asserts that the alternating sequence $\{w_k\}$ is unimodal in absolute value for graphic geometries, and empirical evidence suggests this may hold in general. An inequality for $w_r = \mu(0, 1)$ in terms of the point-set-partition induced by a maximal chain in L appears in [7]. We establish here (Theorem 2) a lower bound on $(-1)^k w_k$ in terms of k, r, and n, and show that equality holds when $k \geq 2$ only for the direct product of a line and a free geometry.

Our results are stated in §3 following a brief section (§2) on preliminaries. In §4 we verify that equality holds in Theorems 1 and 2 for the lattices described. The proofs of these theorems appear in §5 and §6, respectively.

2. **Preliminaries.** Definitions and results required in the sequel are summarized in this section. A detailed treatment of geometric lattices may be found in [2] or [4].

Let L be a finite lattice. If $x \leq y$, the *interval* $[x, y]$ of L is the sublattice $[x, y] = \{z \mid x \leq z \leq y\}$. An element y *covers* x iff $x < y$ and $x < z \leq y$ implies $z = y$, thus $[x, y] = \{x, y\}$. A *point* of L is an element covering 0, the minimum element of L. A *copoint* of L is an element covered by 1, the maximum element of L. A *chain* of length k in L is a linearly-ordered subset $\{x_i \mid x_0 < x_1 < \cdots < x_k\}$ of $k+1$ elements. A *maximal chain* in $[x, y]$ is a chain $\{x_i \mid x_0 < x_1 < \cdots < x_k\}$ such that $x_0 = x$, $x_k = y$, and x_i covers x_{i-1}, $1 \leq i \leq k$. L satisfies the *Jordan-Dedekind chain condition* iff all maximal chains in any interval $[x, y]$ are of equal length. In this case the *rank* $\rho(x)$ of $x \in L$ is the length of a maximal chain in $[0, x]$. The *rank* of L is the rank of its unit element 1.

A finite lattice L is *geometric* when y covers x is equivalent to $y = x \vee p$ for some point $p \not\leq x$. Equivalently, the elements covering an element $x < 1$ partition the set of points $\not\leq x$. The Jordan-Dedekind chain condition holds in a geometric lattice, and its rank function satisfies the *semimodular inequality* $\rho(x \vee y) + \rho(x \wedge y) \leq \rho(x) + \rho(y)$. L is *modular* if equality holds for all x, y. If L is geometric of rank r, elements of rank 1, 2, 3, $r-1$ are *points*, *lines*, *planes*, *copoints*, respectively. Every interval of a geometric lattice is geometric, and direct products of (modular) geometric lattices are (modular) geometric lattices.

Geometric lattices are the order-theoretic counterparts of *combinatorial geometries* [4], or *matroids*, the elements of the lattice representing the closed subsets of points of the geometry, ordered by inclusion. We shall employ geometrical language where convenient in arguments below.

A *separator* of a geometric lattice L is an element $x \neq 0, 1$ such that $L \cong [0, x] \times [x, 1]$. If x is a separator, then so is $y = \bigvee \{p | p \text{ a point } \nleq x\}$, and the mapping $z \mapsto z \vee x$ is an isomorphism between $[0, y]$ and $[x, 1]$. L is *connected* if it has no separators. An *isthmus* is a separator which is a point of L. Then p is an isthmus of L iff there is a copoint c such that $q \leq c$ for every point $q \neq p$.

The *truncation* of a geometric lattice L of rank r is the subset $\{x \in L | \rho(x) \neq r - 1\}$ in the induced order, a geometric lattice of rank $r - 1$. By a sequence of $r - k$ truncations L may be reduced to a geometric lattice L' of rank k whose copoints are the elements of rank $k - 1$ in L. We call L' the truncation of L to rank k.

The *free geometry* (boolean algebra) with j points is the geometric lattice (of rank j) in which every point is an isthmus, and is isomorphic to the lattice of all subsets of its point set under the inclusion order. A j-point *line* is a geometric lattice of rank two with j points. A j-point *projective plane* is a connected, modular geometric lattice of rank three with j points. Each of these three types of lattices is modular, hence so are direct products of them.

The *Möbius function* [10] of a finite lattice L is the function $\mu: L \times L \to \mathbf{Z}$ defined recursively by $\mu(x, y) = 0$ if $x \nleq y$, $\mu(x, y) = 1$ if $x = y$, and $\mu(x, y) = -\Sigma\{\mu(x, z) | x \leq z < y\}$ if $x < y$.

3. Main results. Let L be a finite geometric lattice of rank r with n points and rank function ρ. The *Whitney numbers* of L, *of the second kind*, are the integers

$$W_k = |\{x \in L | \rho(x) = k\}|, \qquad 0 \leq k \leq r. \tag{3.1}$$

Thus $W_0 = W_r = 1$ and $W_1 = n$ by definition. Basterfield and Kelly [1] and Greene [6] proved the inequality

$$W_k \geq n, \qquad 1 \leq k \leq r - 1. \tag{3.2}$$

Greene showed further that equality holds in (3.2) for some k, $2 \leq k \leq r - 1$, iff $k = r - 1$ and L is modular. If $1 \leq k < r - 1$, (3.2) follows immediately from

$$W_{r-1} \geq n, \tag{3.3}$$

on application of (3.3) to the truncation of L to rank $k + 1$. Inequality (3.2) is strengthened substantially when $2 \leq k \leq r - 2$ by

Theorem 1. *Let L be a finite geometric lattice of rank r with n points. Then*

$$W_k \geq \binom{r-2}{k-1}(n - r) + \binom{r}{k}, \qquad 0 \leq k \leq r. \tag{3.4}$$

When $r \geq 4$, equality holds in (3.4) for some k, $2 \leq k \leq r-2$, iff L is (isomorphic to) the direct product of a modular plane and a free geometry.

By Greene's result, the latter conclusion is valid also when equality holds in (3.4) for $k = 2$, $r = 3$, in which case the free geometry is trivial (rank zero).

The extremal lattices may be described in greater detail. A modular plane is either projective, or if not connected, the direct product of a line and a one-point free geometry. Denote by F_j, Q_j, P_j a j-point free geometry, line, and (arbitrary) projective plane, respectively, and let

$$Q(i, j) = Q_i \times F_j, \tag{3.5}$$

$$P(i, j) = P_i \times F_j. \tag{3.6}$$

Then the conclusion when equality holds in (3.4) may be stated:

$$L \cong Q(n - r + 2, r - 2) \quad \text{or} \quad L \cong P(n - r + 3, r - 3).$$

The *Whitney numbers of the first kind* are the integers

$$w_k = \sum_{\rho(x)=k} \mu(0, x), \qquad 0 \leq k \leq r, \tag{3.7}$$

μ being the Möbius function of L. Since $\mu(x, y)$ is nonzero when $x \leq y$, with sign $(-1)^{\rho(y)-\rho(x)}$ [10], w_k is nonzero with sign $(-1)^k$. Thus

$$w_k^+ = (-1)^k w_k \tag{3.8}$$

is positive. Trivially, $w_0^+ = 1$ and $w_1^+ = n$.

Theorem 2. *Let L be a finite geometric lattice of rank r with n points. Then*

$$w_k^+ \geq \binom{r-1}{k-1}(n-r) + \binom{r}{k}, \qquad 0 \leq k \leq r. \tag{3.9}$$

Equality holds in (3.9) for some k, $2 \leq k \leq r$, iff L is isomorphic to the direct product of a line and a free geometry.

Thus $L \cong Q(n - r + 2, r - 2)$ when equality holds.

4. The extremal lattices. In this section we verify that equality holds in (3.4) for $Q(n - r + 2, r - 2)$ and $P(n - r + 3, r - 3)$ and in (3.9) for $Q(n - r + 2, r - 2)$. In the proof of Theorem 2 (§6), we shall require the fact that (3.9) is a strict inequality when $k \geq 2$ for $P(n - r + 3, r - 3)$, a result most conveniently established at this point.

In computing the Whitney numbers of a direct product, it is useful to consider the polynomials

$$\phi(\lambda) = \sum_{x \in L} \lambda^{r-\rho(x)}, \tag{4.1}$$

$$\chi(\lambda) = \sum_{x \in L} \mu(0, x)\lambda^{r-\rho(x)}, \tag{4.2}$$

where r is the rank of L. Thus the coefficient of λ^{r-k} is W_k in (4.1) and w_k in (4.2). The latter is the *characteristic polynomial* of L.

For a direct product $L = L_1 \times L_2$ of geometric lattices, $\rho((x, y)) = \rho_1(x) + \rho_2(y)$ and $\mu((0, 0), (x, y)) = \mu_1(0, x)\mu_2(0, y)$ [10]; thus

$$\phi(\lambda) = \phi_1(\lambda)\phi_2(\lambda), \tag{4.3}$$

$$\chi(\lambda) = \chi_1(\lambda)\chi_2(\lambda). \tag{4.4}$$

The polynomials $\phi(\lambda)$, $\chi(\lambda)$ are well known for a free geometry, line, and projective plane, and are given below in Table 1. Since the existence of P_j implies $j = s^2 + s + 1$ for some integer $s \geq 2$, we make this substitution where convenient.

	$\phi(\lambda)$	$\chi(\lambda)$
F_j	$\sum_{i=0}^{j} \binom{j}{i}\lambda^{j-i}$	$\sum_{i=0}^{j} (-1)^i\binom{j}{i}\lambda^{j-i}$
Q_j	$\lambda^2 + j\lambda + 1$	$\lambda^2 - j\lambda + (j - 1)$
P_j	$\lambda^3 + j\lambda^2 + j\lambda + 1$	$\lambda^3 - (s^2 + s + 1)\lambda^2 + s(s^2 + s + 1)\lambda - s^3$, $j = s^2 + s + 1$

Table 1

From (4.3) and Table 1 we obtain the $\phi(\lambda)$-coefficient of λ^{r-k} for $Q(n - r + 2, r - 2)$, $P(n - r + 3, r - 3)$, respectively, as

$$(4.4)\qquad \binom{r-2}{k} + \binom{r-2}{k-1}(n-r+2) + \binom{r-2}{k-2},$$

$$(4.5)\qquad \binom{r-3}{k} + \binom{r-3}{k-1}(n-r+3) + \binom{r-3}{k-2}(n-r+3) + \binom{r-3}{k-3}.$$

Using Pascal's identity, both (4.4) and (4.5) reduce to the right-hand side of (3.4).

From (4.4) and Table 1 we obtain the $\chi(\lambda)$-coefficient of $(-1)^k\lambda^{r-k}$ for $Q(n-r+2, r-2)$ as

$$(4.6)\qquad \binom{r-2}{k} + (n-r+2)\binom{r-2}{k-1} + (n-r+1)\binom{r-2}{k-2},$$

which simplifies to the right-hand side of (3.9), and for $P(n-r+3, r-3)$ as

$$(4.7)\quad \binom{r-3}{k} + (s^2+s+1)\binom{r-3}{k-1} + s(s^2+s+1)\binom{r-3}{k-2} + s^3\binom{r-3}{k-3}.$$

Setting $n-r+3 = s^2+s+1$ and using Pascal's identity, we can rewrite (4.6) as

$$(4.8)\ \binom{r-3}{k} + (s^2+s+1)\binom{r-3}{k-1} + 2(s^2+s-\tfrac{1}{2})\binom{r-3}{k-2} + (s^2+s-1)\binom{r-3}{k-3}.$$

The first two terms in (4.7) and (4.8) are equal, but if $2 \le k \le r$, at least one of $\binom{r-3}{k-2}$, $\binom{r-3}{k-3}$ is positive. Since $s \ge 2$, $s(s^2+s+1) > 2(s^2+s-\tfrac{1}{2})$ and $s^3 > s^2+s-1$, hence (4.7) is strictly greater than (4.8) when $2 \le k \le r$.

5. Proof of Theorem 1. We proceed now to the proof of inequality (3.4). The following notation will be used, for an interval $[u, v]$ of L.

$$A_j(u, v) = \{x \in [u, v] \mid \rho(x) = j\}, \qquad \alpha_j(u, v) = |A_j(u, v)|.$$

$$B_j(u, v) = \{x \notin [u, v] \mid \rho(x) = j\}, \qquad \beta_j(u, v) = |B_j(u, v)|.$$

Thus $W_j = \alpha_j(u, v) + \beta_j(u, v)$. Since $A_j(u, v)$ is the set of elements of rank $j - \rho(u)$ in the interval $[u, v]$, we have

$$(5.1)\qquad W_{j-\rho(u)}([u, v]) = \alpha_j(u, v).$$

We shall require the following lemma. A proof is given in our related paper [5, Corollary to Theorem 4].

Lemma. *For any point p of L,*

$$(5.2)\qquad \alpha_2(p, 1) + \beta_{r-1}(p, 1) \ge \alpha_1(0, 1).$$

The proof of (3.4) will be by induction on the sum $r + k$. When $r + k \le 5$, the only nontrivial case is $r = 3$, $k = 2$, where (3.4) follows from (5.2) (or from (3.3)). As the inductive hypothesis we assume that if L' is a geometric lattice of rank r' with n' points, then (3.4) holds for all k' such that $r' + k' < r + k$, where $r + k \ge 6$. By (3.3) we may assume $2 \le k \le r - 2$, so $r \ge 4$.

Fix a point p of L. Then

$$W_k = \alpha_k(p, 1) + \beta_k(p, 1). \tag{5.3}$$

Let $l = \alpha_2(p, 1) = W_1([p, 1])$. The interval $[p, 1]$ is of rank $r - 1$, so by (5.1) and the inductive hypothesis,

$$\alpha_k(p, 1) \ge \binom{r-3}{k-2}(l - r + 1) + \binom{r-1}{k-1}. \tag{5.4}$$

To obtain a lower bound on $\beta_k(p, 1)$, we first observe that $y \in A_{k+1}(p, 1)$ iff $y = x \vee p$ for some $x \in B_k(p, 1)$. The mapping $x \mapsto x \vee p$ is thus a surjection $B_k(p, 1) \to A_{k+1}(p, 1)$, and so partitions $B_k(p, 1)$ into inverse images of elements of $A_{k+1}(p, 1)$. The inverse image of $y \in A_{k+1}(p, 1)$ is the subset $A_k(0, y) - A_k(p, y)$ of $B_k(p, 1)$, of cardinality $\alpha_k(0, y) - \alpha_k(p, y)$. By (5.2), $\alpha_k(0, y) - \alpha_k(p, y) \ge \alpha_1(0, y) - \alpha_2(p, y)$. Thus

$$\beta_k(p, 1) \ge \sum_{y \in A_{k+1}(p,1)} \alpha_1(0, y) - \sum_{y \in A_{k+1}(p,1)} \alpha_2(p, y). \tag{5.5}$$

The two sums in (5.5) may be written as follows on interchanging the order of summation in each.

$$\begin{aligned} \sum_{y \in A_{k+1}(p,1)} \alpha_1(0, y) &= \alpha_{k+1}(p, 1) + \sum_{q \in B_1(p,1)} \alpha_{k+1}(p \vee q, 1) \\ &= \alpha_{k+1}(p, 1) + \sum_{a \in A_2(p,1)} (\alpha_1(0, a) - 1)\alpha_{k+1}(a, 1). \end{aligned} \tag{5.6}$$

$$\sum_{y \in A_{k+1}(p,1)} \alpha_2(p, y) = \sum_{a \in A_2(p,1)} \alpha_{k+1}(a, 1). \tag{5.7}$$

Substituting (5.6) and (5.7) into (5.5) gives

$$\beta_k(p, 1) \ge \alpha_{k+1}(p, 1) + \sum_{a \in A_2(p,1)} (\alpha_1(0, a) - 2)\alpha_{k+1}(a, 1). \tag{5.8}$$

Writing $\alpha_{k+1}(a, 1) = \alpha_{k+1}(a, 1) - \binom{r-2}{k-1} + \binom{r-2}{k-1}$ in (5.8), and noting that

$\Sigma_{a \in A_2(p,1)}(\alpha_1(0, a) - 2) = n - 1 - l$, we obtain

$$\beta_k(p, 1) \geq \alpha_{k+1}(p, 1) + \binom{r-2}{k-1}(n - 1 - l) + C_p, \tag{5.9}$$

where

$$C_p = \sum_{a \in A_2(p,1)} (\alpha_1(0, a) - 2)\left(\alpha_{k+1}(a, 1) - \binom{r-2}{k-1}\right) \geq 0, \tag{5.10}$$

since $\alpha_1(0, a) = W_1([0, a]) \geq 2$ and $\alpha_{k+1}(a, 1) = W_{k-1}([a, 1]) \geq \binom{r-2}{k-1}$. The interval $[a, 1]$ is of rank $r - 2$, and $2 \leq k \leq r - 2$ implies $1 \leq k - 1 \leq (r - 2) - 1$, so $W_{k-1}([a, 1]) = \binom{r-2}{k-1}$ iff $[a, 1] \cong F_{r-2}$. Thus equality holds in (5.10) iff for every line a on p, either $[0, a] \cong F_2$ or $[a, 1] \cong F_{r-2}$.

The interval $[p, 1]$ is of rank $r - 1$, so by (5.1) and the inductive hypothesis

$$\alpha_{k+1}(p, 1) \geq \binom{r-3}{k-1}(l - r + 1) + \binom{r-1}{k}. \tag{5.11}$$

We obtain finally, from (5.3), (5.4), (5.9), (5.10), and (5.11),

$$\begin{aligned} W_k &\geq \binom{r-3}{k-2}(l - r + 1) + \binom{r-1}{k-1} + \binom{r-3}{k-1}(l - r + 1) \\ &\quad + \binom{r-1}{k} + \binom{r-2}{k-1}(n - 1 - l) \\ &= \binom{r-2}{k-1}(n - r) + \binom{r}{k}. \end{aligned}$$

Suppose now that $2 \leq k \leq r - 2$ and equality holds in (3.4). Then equality holds in (5.4) and (5.11) for every point p of L, where $l = \alpha_2(p, 1)$, and equality holds in (5.10) for every line a of L. To prove that $L \cong Q(n - r + 2, r - 2)$ or $L \cong P(n - r + 3, r - 3)$, we again argue by induction on the sum $r + k$. Thus assume that if L' is a geometric lattice of rank r' with n' points, and equality holds in (3.4) for some k', $2 \leq k' \leq r' - 2$, then $L' \cong Q(n' - r' + 2, r' - 2)$ or $L' \cong Q(n' - r' + 2, r' - 3)$, whenever $r' + k' < r + k$. The initial case is $r + k = 6$, when $r = 4$, $k = 2$. As the proof for this case is similar to the inductive step, it will be convenient to postpone it. Thus we assume $r + k \geq 7$, so $r \geq 5$. Then at least one of the pairs $(r', k') = (r - 1, k - 1)$, $(r - 1, k)$ satisfies $2 \leq k' \leq r' - 2$, so by the inductive hypothesis, for every point p of L, either

(5.12) $$[p, 1] \cong Q(l - r + 3, r - 3),$$

or

(5.13) $$[p, 1] \cong P(l - r + 4, r - 4),$$

where $l = \alpha_2(p, 1)$ is the number of lines of L on p. Also, from equality in (5.10) we have for every line a of L, either $[0, a] \cong F_2$ or $[a, 1] \cong F_{r-2}$.

Suppose first that every line of L has two points ($[0, a] \cong F_2$). Let p be a point of L. Since $r \geq 5$, (5.12)–(5.13) imply that $[p, 1]$ has an isthmus b. Then in L, b is the only line on p not on some copoint c on p. The second point q on a is thus the only point of L not on c, so q is an isthmus of L. The number of lines on q is $n - 1$, so by (5.12)–(5.13),

(5.14) $$L \cong [0, c] \times [0, q] \cong [p, 1] \times F_1 \cong \begin{cases} Q(n - r + 2, r - 2) \\ \text{or} \\ P(n - r + 3, r - 3). \end{cases}$$

Suppose now that there is a line a of L with at least three points. Then $[a, 1] \cong F_{r-2}$. Let $u_1, u_2, \cdots, u_{r-2}$ be the planes on a. The join of any set of j of these is an element of rank $j + 2$ in L. Let p be a point of a and suppose first that (5.13) holds. Then there is an element x of rank four in L such that $x > p$ and $[p, x]$ is a projective plane. There are then two points p_1, p_2 of x on different planes through a, say u_1, u_2, respectively. The plane $p \vee p_1 \vee p_2$ of x intersects u_i, $3 \leq i \leq r - 2$, only in p, as otherwise $u_1 \vee u_2 \vee u_i$ is of rank four. But then x contains only two lines $p \vee p_1$, $p \vee p_2$ on p, contradicting that $[p, x]$ is a projective plane, hence has no two-point lines. Thus we may assume that (5.12) holds for every point p on a, so at most one of the planes on a contains more than one line on p other than a. Suppose u_1, u_2 each contain two points off a, say p_1, q_1 in u_1 and p_2, q_2 in u_2. Since a has at least three points, there is a point p of a not on either of the lines $p_1 \vee q_1$, $p_2 \vee q_2$. We then have three lines a, $p \vee p_i$, $p \vee q_i$ on p in u_i, $i = 1, 2$, a contradiction. Thus u_1, say, has only a single point q off a. Then q is the only point of L off the copoint $c = u_2 \vee \cdots \vee u_{r-2}$, hence is an isthmus of L. The number of lines on q is $l = n - 1$, and the argument preceding (5.14) can be repeated.

It remains only to verify the result for the case $r = 4$, $k = 2$. From equality in (5.11) we have, by Greene's result, that $[p, 1]$ is a modular plane for every point p of L. If there is a nontrivial line a in L, the argument above gives an isthmus q on one of the two planes through a, and we are finished as before. If every line has two points, then by equality in (3.4) we have $W_2 = \binom{n}{2} = 2n - 2$, which implies $n = r = 4$, so $L \cong F_4 \cong Q(2, 2)$.

6. Proof of Theorem 2. We consider first the case $k = r$ in Theorem 2. Let $\mu^+(0, x) = (-1)^{\rho(x)}\mu(0, x)$, a positive integer. If p is a point of L, we have by Weisner's theorem [10],

$$\mu^+(0, 1) = \sum \mu^+(0, c), \tag{6.1}$$

where the sum is over all copoints c of L such that $c \not> p$.

Proposition 1. *If L is a geometric lattice of rank r with n points, then*

$$\mu^+(0, 1) \geq n - r + 1, \tag{6.2}$$

with equality iff $L \cong Q(n - r + 2, r - 2)$.

Proof. The proof is by induction on r. Equality holds in (6.2) for $r = 2$, so assume inductively that (6.2) holds for $r' < r$, where $r \geq 3$. If c is a copoint of L, let $\alpha(c)$ denote the number of points in $[0, c]$. By the inductive hypothesis,

$$\mu^+(0, c) \geq \alpha(c) - r + 2. \tag{6.3}$$

Summing (6.2) over all points p of L, and using (6.3), we obtain

$$n\mu^+(0, 1) \geq \sum_{c \text{ copoint}} (n - \alpha(c))(\alpha(c) - r + 2). \tag{6.4}$$

But for any copoint c, $r - 1 \leq \alpha(c) \leq n - 1$, so

$$\begin{aligned}(n - \alpha(c))(\alpha(c) - r + 2) &= (n - 1 - \alpha(c))(\alpha(c) - (r - 1)) + n - r + 1 \\ &\geq n - r + 1,\end{aligned} \tag{6.5}$$

with equality iff $\alpha(c) = r - 1$ or $n - 1$. Thus from (6.4), (6.5), and (3.3)

$$\mu^+(0, 1) \geq W_{r-1}(n - r + 1)/n \geq n - r + 1. \tag{6.6}$$

To show that equality in (6.2) implies $L \cong Q(n - r + 2, r - 2)$, we again argue by induction on r. The result is trivially true if $r = 2$, so assume $r \geq 3$, and that equality holds in (6.2). By (6.6), $W_{r-1} = n$, and from (6.5) every copoint of L has either $r - 1$ or $n - 1$ points. If all copoints have $r - 1$ points, then either $L \cong F_r \cong Q(2, r - 2)$, or else L is the truncation to rank r of a free geometry F_n, $n > r$. But in the latter case, $W_{r-1} = \binom{n}{r-1} > n$. If some copoint c contains $n - 1$ points, the point p not on c is an isthmus of L. Since equality must hold in (6.3) for every copoint, $[0, c] \cong Q(n - r + 2, r - 3)$. Then $L \cong [0, c] \times [0, p] \cong Q(n - r + 2, r - 2)$, and the lemma is proved.

Let $A_i = \{x \in L \mid \rho(x) = i\}$ and for $x \in L$, let $\alpha(x)$ be the number of points in $[0, x]$. Define $m_k^+ = \Sigma_{x \in A_k} \alpha(x)\mu^+(0, x)$. We then have

Proposition 2. *Let L be a geometric lattice of rank r with n points. Then*

$$m_k^+ \geq (n - r + 1)W_{k-1} + (r - k)w_{k-1}^+, \qquad 1 \leq k \leq r. \tag{6.7}$$

Proof. The case $k = 1$ is trivial, and the first inequality in (6.6) is the case $k = r$, so assume $2 \leq k \leq r - 1$. Then by applying (6.1) to the interval $[0, x]$ below, we obtain

$$\begin{aligned} m_k^+ &= \sum_{x \in A_k} \sum_{p \in A_1, x > p} \mu^+(0, x) = \sum_{p \in A_1} \sum_{x \in A_k, x > p} \mu^+(0, x) \\ &= \sum_{p \in A_1} \sum_{x \in A_k, x > p} \left(\sum_{y \in A_{k-1}, p \nleq y < x} \mu^+(0, y) \right) \\ &= \sum_{y \in A_{k-1}} \mu^+(0, y) \sum_{p \in A_1, p \nleq y} \sum_{x \in A_k, x = p \vee y} 1 \\ &= \sum_{y \in A_{k-1}} \mu^+(0, y)(n - \alpha(y)) \\ &= \sum_{y \in A_{k-1}} \mu^+(0, y)(n - \alpha(y) - r + k) + (r - k) \sum_{y \in A_{k-1}} \mu^+(0, y). \end{aligned}$$

We next apply (6.2) to the interval $[0, y]$.

$$\begin{aligned} m_k^+ &\geq \sum_{y \in A_{k-1}} (\alpha(y) - k + 2)(n - \alpha(y) - r + k) + (r - k)w_k^+ \\ &\geq (n - r + 1)W_{k-1} + (r - k)w_k^+, \end{aligned}$$

the last inequality following since $k - 1 \leq \alpha(y) \leq n - r + k - 1$ for y of rank $k - 1$, so

$$\begin{aligned} &(\alpha(y) - k + 2)(n - \alpha(y) - r + k) \\ &\qquad = (\alpha(y) - k + 1)(n - r + k - 1 - \alpha(y)) + (n - r + 1) \\ &\qquad \geq n - r + 1. \end{aligned}$$

We consider now inequality (3.9), arguing by induction on k. The case $k = 1$ is trivial, and by (6.2) we may assume $k < r$, so let $2 \leq k \leq r - 1$ and suppose (3.9) holds for all $k' < k$.

Let p be a point of L. Applying (6.1) to $[0, y]$, for $y \in A_{k+1}$, $y > p$, we obtain

$$w_k^+ = \sum_{x\in A_k} \mu^+(0, x) = \sum_{x\in A_k, x>p} \mu^+(0, x) + \sum_{x\in A_k, x\not>p} \mu^+(0, x)$$

$$= \sum_{x\in A_k, x>p} \mu^+(0, x) + \sum_{y\in A_{k+1}, y>p} \mu^+(0, y).$$

Summing over all points p, and using (6.7),

$$nw_k^+ = \sum_{x\in A_k} \alpha(x)\mu^+(0, x) + \sum_{y\in A_{k+1}} \alpha(y)\mu^+(0, y) = m_k^+ + m_{k+1}^+$$

$$\geq (n - r + 1)(W_{k-1} + W_k) + (r - k)w_{k-1}^+ + (r - k - 1)w_k^+.$$

We now apply (3.4) and the inductive hypothesis, obtaining

$$(6.8)\quad \begin{aligned} (n - r + 1 + k)w_k^+ &\geq (n - r + 1)(W_{k-1} + W_k) + (r - k)w_{k-1}^+ \\ &\geq (n - r + 1)\left\{\binom{r-2}{k-2}(n - r) + \binom{r}{k-1} + \binom{r-2}{k-1}(n - r) + \binom{r}{k}\right\} \\ &\quad + (r - k)\left\{\binom{r-1}{k-2}(n - r) + \binom{r}{k-1}\right\} \\ &= (n - r + 1)\left\{\binom{r-1}{k-1}(n - r) + \binom{r}{k}\right\} + k\left\{\binom{r-1}{k-1}(n - r) + \binom{r}{k}\right\} \end{aligned}$$

after simplification. Thus

$$(n - r + 1 + k)w_k^+ \geq (n - r + 1 + k)\left\{\binom{r-1}{k-1}(n - r) + \binom{r}{k}\right\},$$

so (3.9) follows.

To complete the proof of Theorem 2, assume equality holds in (3.9) for some k, $2 \leq k \leq r$. If $k = r$, the result follows from Proposition 1, so suppose $2 \leq k \leq r - 1$. The proof is by induction on r. Consider first the case $r = 3$, $k = 2$. From (6.8), equality in (3.9) implies $W_2 = n$, so L is a modular plane, and therefore $L \cong Q(n - 1, 1)$ or $L \cong P(n, 0)$. But the latter is impossible (§4), hence $L \cong Q(n - 1, 1)$.

Assume inductively that the result holds for $r' < r$, where $r \geq 4$. Then equality in (3.9) implies equality in (6.8), so W_k, W_{k-1} attain the lower bound in (3.4). At least one of $k - 1$, k satisfy $2 \leq k' \leq r - 2$, so by Theorem 1, either $L \cong Q(n - r + 2, r - 2)$ or $L \cong P(n - r + 3, r - 3)$. But $k \geq 2$, so again the latter is impossible, and the proof is complete.

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